Monte Carlo Simulation for BGM

Brace-Gatarek-Musiela (BGM) model, also called LIBOR Market Model, is a multi-factor log-normal model for pricing interest rate derivatives. The model is usually solved by Monte Carlo simulation.

Diffusion is performed over a period $[0, t_{max}]$. This period is divided into time steps $0 = t_0 < t_1 < \ldots < t_N = t_{max}$. The dates t_i are called *Diffusion Dates.* They may or may not be regularly spaced. The interval between two consecutive diffusion dates $[t_i, t_{i+1}]$ is called a *Diffusion Step.*

The model is aimed to calibrate its parameters on the market price of a list of at-the-money FX options, with increasing maturities $\tau_1 < \ldots < \tau_m \leq t_{max}$, which we call *Calibration Dates.* An interval $[\tau_i, \tau_{i+1}]$ is a *Calibration Interval* or a *Slice.*

Diffusion dates are set in such a way that all calibration dates are diffusion dates. Then the diffusion step is specified on each slice. Diffusion dates are business days, approximately spaced by the diffusion step (except the last step that can be shorter). We define the indices $i_1, \ldots, i_m$ by $t_{i_j} = \tau_j$, $j = 1, \ldots, m$.

Calibration dates and diffusion dates are dates at which the FX and interest rate values are observed. Corresponding *Spot Dates* are denoted t_i' and τ_i'.

Generally speaking, all diffusions will be considered between diffusion dates, whereas discounting will be performed between spot dates.

Forward Libor rates are settled at fixed dates $T_1, \ldots, T_p$, called *Future Dates,* which do not depend on the diffusion dates t_k and are regularly spaced by the BGM tenor d (usually 3 months): $T_{i+1} = T_i + d$ (up to non business days).

For a given diffusion date t_k and corresponding spot date t'_k there is a unique index i_k such that:

$$T_{i_k - 1} \leq t'_k < T_{i_k}$$

The interval $\lfloor t_k , \tau_{i_k} \rfloor$ is called the *Stub Interval* and the rate that applies on this period is the *Stub Rate.*

At each diffusion date t_k, the *Spot Libor Rate* applies on the interval $[t'_k, t'_k + d]$ (or, to be more precise, until the first business day greater than or equal to $t'_k + d$). Spot Libor rate is not used in the diffusion procedure of discount factor of other maturities. It is only displayed to price exotic options with a barrier on the Libor, or similar structures.

It is important to notice that these dates should be absolutely the same in both currencies. In particular, the spot dates are computed in the same way as those of an FX option.

At a given diffusion date t_k, the curve structure is constituted of several rates (ref. https://finpricing.com/lib/IrCurveIntroduction.html):

- Short rate r_k on $[t'_k, t'_{k+1}]$
- Stub rate s_k on $[t'_k, T_{i_k+1}]$
- Spot Libor L_k on $[t'_k, t'_k + d]$
- Future Libor F_{ki} on $[T_i, T_{i+1}], \quad i \geq i_k$

Discount factors are computed as follows:

- $ShortDF_k = DF(t'_k, t'_{k+1}) = \dfrac{1}{1+(t'_{k+1} - t'_k)\, r_k}$
- $StubDF_k = DF(t'_k, T_{i_k}) = \dfrac{1}{1+(T_{i_k} - t'_k)\, s_k}$
- $SpotLiborDF_k = DF(t'_k, t'_k + d) = \dfrac{1}{1+d.L_k}$
- By induction: $FutureDF_{i+1} = DF(t'_k, T_{i+1}) = \dfrac{DF(t'_k, T_i)}{1+(T_{i+1} - T_i)\, F_{ki}}$

The *numeraire* is the product of short discount factors:

$$Num_k = Num(t'_k) = \prod_{h=0}^{k-1} DF(t'_h, t'_{h+1})$$

It is known at date t_{k-1} (and is deterministic if $k = 1$).

This rate structure is duplicated at each diffusion date: one structure for the domestic currency and the other for the foreign one, specified by superscripts d and f .

When computed at a diffusion date other than t_0 , these rates and discount factors are random variables that are materialized as Monte-Carlo arrays. Each index corresponds to a path and expectations will be replaced by averages over all paths.

At each diffusion date, only a certain number of discount factors are provided. For the purpose of rate diffusion, one may need other discount factors. An interpolation procedure is thus necessary. Currently, we have implemented a linear interpolation of the logarithm of encompassing available maturities. Assume for instance that these maturities are in the following order:

$$t'_k < t'_{k+1} < T_{i_k} < t'_k + d < T_{i_k+1} < T_{i_k+2} < \dots$$

(the position of t'_{k+1} depends on the diffusion step and on the stub period). Then, for any maturity date $T > t'_k$, we check encompassing maturities, say $T_i < T < T_{i+1}$, and we set:

$$DF(t'_k, T) = DF(t'_k, T_i)^{\frac{T_{i+1}-T}{T_{i+1}-T_i}} \, DF(t'_k, T_{i+1})^{\frac{T-T_i}{T_{i+1}-T_i}}$$

This interpolation, which is very commonly used, means that forward spot rates are piecewise constant. It is similar to, but not quite the same as making a linear interpolation of the yields to maturity. It has been chosen because accuracy is the same and formulas are simpler.

Note that $DF(t'_k, t'_k) = 1$ and $DF(t'_k, T)$ is not available if $T < t'_k$.

Forward discount factors are simply ratios of spot ones:

$$DF_{\text{forward}}(t_k; T, T+\delta) = \frac{DF(t'_k, T+\delta)}{DF(t'_k, T)}$$

In the sequel, forward rates (Libor and CMS rates) are computed with this interpolation technique.

The curve diffusion procedure computes the rates at date t_{k+1} with respect to t_k. First, one computes the forward Libor rate starting at t'_{k+1} from the DF function:

$$F(t_k; t'_{k+1}, t'_{k+1} + d) = \frac{1}{d}\left(\frac{DF(t'_k, t'_{k+1})}{DF(t'_k, t'_{k+1} + d)} - 1\right)$$

Then the BGM diffusion equation is discretized to get the spot Libor rate L_{k+1} from its forward value $F(t_k; t'_{k+1}, t'_{k+1} + d)$ and the future rates F_{k+1i} from F_{ki}, $i \geq i_{k+1}$. Let:

$$\delta t = t_{k+1} - t_k$$

and $dW_1,\ldots,dW_4$ be independent Gaussian random variables with standard deviation $\sqrt{\delta t}$ (in practice, arrays of real numbers indexed by the Monte-Carlo path index). We first write:

$$L_{k+1} = F(t_k;t'_{k+1},d)\ \exp\left(\sigma(t_k,t'_{k+1})\sum_{j=1}^{4}\psi_j(0)\left(dW_j - \tfrac{1}{2}\sigma(t_k,t'_{k+1})\,\delta t\,\psi_j(0)\right)\right)$$

$$F_{k+1i} = F_{ki}\ \exp\left(\sigma(t_k,T_i)\sum_{j=1}^{4}\psi_j(T_i - t'_{k+1})\left(dW_j - \tfrac{1}{2}\sigma(t_k,T_i)\,\delta t\,\psi_j(T_i - t'_{k+1})\right)\right)$$

The drift is here stated to make L_{k+1} and other future rates have their forward value as expectation. This is slightly wrong because of an Ito term, due to a discrepancy between the risk-neutral probability and the forward neutral one, but the error is smaller than the Monte-Carlo error and will anyway be adjusted as explained in sect. II.9 to make actualized expectation of discount factors and numeraires match exactly the initial discount factors.

The volatility triangle is an array that reflects the function $\sigma(t,T)$. However, it is provided by the HJM sheet, with a different mesh than our original mesh $(t_k,T_i)_{k,i}$. Therefore, $\sigma(t_k,T_i)$ is interpolated from values in the volatility triangle, and so is $\sigma(t_k,T)$ when T itself is interpolated.

The volatility $\sigma(t_k,T)$ applies on the period $[t_k,t_{k+1}]$ to a rate that corresponds to a loan on the period $[T,T+d]$, with $T \geq t'_{k+1}$ (the spot date corresponding to t_{k+1}). The date T can be either one of the future dates T_i and then $T+d = T_{i+1}$ or it can be any interpolated date. Let $(\tilde{t}_h,\tilde{T}_i)_{h,i}$ be the HJM mesh (typically both $\tilde{t}_h$ and $\tilde{T}_i$ are 3 months spaced and $T_i = t'_i$). We assume that for a given date t, $\sigma(t,T)$ is linearly interpolated between encompassing maturities $[\tilde{T}_i,\tilde{T}_{i+1}]$ and that, as a function of t, it is constant on intervals $[\tilde{t}_h, \tilde{t}_{h+1}]$, which implies that the <u>square</u> of the volatility $\sigma(t,T)^2$ is linearly interpolated with respect to t.

Suppose for instance that $T = \lambda\tilde{T}_i + (1-\lambda)\tilde{T}_{i+1}$ and $\tilde{t}_h < t_k < \tilde{t}_{h+1} < t_{k+1} < \tilde{t}_{h+2}$. We first set:

$$\sigma(\tilde{t}_h, T) = \lambda\,\sigma(\tilde{t}_h, \tilde{T}_i) + (1-\lambda)\,\sigma(\tilde{t}_h, \tilde{T}_{i+1})$$

$$\sigma(\tilde{t}_{h+1}, T) = \lambda\,\sigma(\tilde{t}_{h+1}, \tilde{T}_i) + (1-\lambda)\,\sigma(\tilde{t}_{h+1}, \tilde{T}_{i+1})$$

Then $\sigma(t_k, T)$ is given by the formula:

$$\sigma(t_k, T)^2 = \frac{1}{\delta t}\Big((\tilde{t}_{h+1} - t_k)\,\sigma(\tilde{t}_h, T)^2 + (t_{k+1} - \tilde{t}_{h+1})\,\sigma(\tilde{t}_{h+1}, T)^2\Big)$$

In the situation where $T < \tilde{t}_{h+2}$, then the second linear interpolation has to be modified, because $\sigma(\tilde{t}_{h+1}, \tilde{T}_i)$ does not exist. We set in this case:

$$\sigma(\tilde{t}_{h+1}, T) = \lambda\,\sigma(\tilde{t}_h, \tilde{T}_i) + (1-\lambda)\,\sigma(\tilde{t}_{h+1}, \tilde{T}_{i+1})$$

If $T = t_{k+1}$, this may be considered as a linear interpolation of the *s*-volatility.